Kernels for linear time invariant system identification

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Abstract

In this report, we focus on the choice of the kernel for identification of a linear time invariant (LTI) dynamical system from the knowledge of the input and a finite set of output observations. We provide guidelines to design the kernel function so as to enforce different types of prior information about the dynamical system under study. On one hand, we characterize general families of kernels that incorporate information such as smoothness, stability, relative degree, absence of oscillatory behavior, or delay. On the other hand, we show that certain popular kernels for curve fitting are not well suited for the identification of stable dynamical systems.

1 Introduction

Identification of LTI systems is a classical and very well studied problem, see e.g. [8]. Although the standard approach is based on parametric models, nonparametric kernel-based techniques have been recently shown to achieve remarkably better performances under a variety of circumstances [4, 9–11], thus renewing the attention on the subject.

In this report, we consider a formulation based on regularization in reproducing kernel Hilbert spaces (RKHS) [1]. The setup is general enough to take into account both discrete and continuous-time linear time-invariant systems, allowing for sparse time sampling of the output signal. For the sake of simplicity, we focus on the case of SISO (single input single output) linear time invariant system identification, where the goal is to reconstruct a scalar impulse response function from the knowledge of the input signal and a finite set of output measurements. Nevertheless, the ideas presented in this report are general enough to be naturally extendable to more complex and structured problems.

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Much effort has been devoted in the literature to develop functional spaces that model certain desirable properties of signals. Somehow surprisingly, considerably less effort has been devoted in designing function spaces for modeling impulse responses of linear dynamical systems. Arguably, the two modeling problems should be handled differently. In this report, we characterize families of RKHS that encode those properties that are specific to impulse responses of dynamical systems, such as causality, stability, absence of oscillations, relative degree, delay, and so on. We build upon a collection of classical results about RKHS, and show how they can be applied to enforce different types of prior knowledge about a dynamical system by simply designing a suitable kernel function.

2 Regularization for LTI system identification

In order to handle both continuous and discrete-time system in a unified framework, we refer to an abstract *time set* $\mathcal{T}$, that is a sub-group of $(\mathbb{R}, +)$. Given an input signal $u : \mathcal{T} \to \mathbb{R}$, a LTI system generates an output signal $y : \mathcal{T} \to \mathbb{R}$ according to the convolution equation

$$y(t) = (h * u)(t) = \int_{\mathcal{T}} u(\tau) h(t - \tau) d\tau,$$

where $h : \mathcal{T} \to \mathbb{R}$ is the *impulse response*, and the integral is taken with respect to the Lebesgue measure. Clearly, if the time set is discrete, the convolution integral reduces to a series.

In the following, we study the problem of identifying the impulse response, assuming availability of the input signal and a finite dataset of output measurement pairs

$$\mathcal{D} = \{(t_1, y_1), \dots, (t_\ell, y_\ell)\}.$$

This is a classical ill-posed deconvolution problem that can be tackled by means of regularization techniques [15, 16]. Consider, for instance, a regularized least squares approach of the type

$$\min_{h \in \mathcal{H}} \left(\sum_{i=1}^{\ell} \frac{(y_i - (h * u)(t_i))^2}{2\lambda} + \frac{\|h\|_{\mathcal{H}}^2}{2} \right), \tag{1}$$

where $\mathcal{H}$ is a Hilbert space of functions, and $\lambda > 0$ is a regularization parameter. Assume that the input signal and the space $\mathcal{H}$ are such that all the pointwise evaluated convolutions are bounded linear functionals, namely, for all $i = 1, \dots, \ell$, there exists a finite constant C_i such that

$$|(h * u)(t_i)| \leq C_i \|h\|_{\mathcal{H}}, \quad \forall h \in \mathcal{H}.$$

Then, by the Riesz representation Theorem [6, 12], there exist unique representers w_i such that

$$(h * u)(t_i) = \langle h, w_i \rangle_{\mathcal{H}}.$$

In addition, one can show that any optimal solution of (1) can be expressed as a linear combination of the representers (a result known as the *representer theorem* [7]):

$$h = \sum_{i=1}^{\ell} c_i w_i. \tag{2}$$

In view of (2), the regularization problem reduces to determining a vector of coefficients c_i of the same dimension of the number of observations. More precisely, an optimal vector of coefficients $c \in \mathbb{R}^\ell$ can be obtained by simply solving a linear system of the form

$$(\mathbf{K} + \lambda \mathbf{I})\, c = y,$$

where $y \in \mathbb{R}^\ell$ denotes the vector of output observations, and the entries of the *kernel matrix* $\mathbf{K}$ are given by

$$\mathbf{K}_{ij} = \langle w_i, w_j \rangle_{\mathcal{H}}.$$

2.1 Reproducing Kernel Hilbert Spaces

Reproducing kernel Hilbert spaces are a family of Hilbert spaces that enjoy particularly favorable properties from the point of view of regularization. As their name suggests, they are strongly linked with the concept of *positive semidefinite kernel*. Given a non-empty set $\mathcal{X}$, a positive semidefinite kernel is a symmetric function $K : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ such that

$$\sum_{i=1}^{\ell} \sum_{j=1}^{\ell} c_i c_j K(x_i, x_j) \geq 0, \qquad \forall (x_i, c_i) \in (\mathcal{X}, \mathbb{R}).$$

A RKHS is a space of functions $h : \mathcal{X} \to \mathbb{R}$ such that point-wise evaluation functionals are bounded. This means that, for any $x \in \mathcal{X}$, there exists a finite constant C_x such that

$$|h(x)| \leq C_x \|h\|_{\mathcal{H}}, \quad \forall h \in \mathcal{H}.$$

Given a RKHS, it can be shown that there exists a unique symmetric and positive semidefinite kernel function K (called the *reproducing kernel*), such that the so-called *reproducing property* holds:

$$h(x) = \langle h, K_x \rangle_{\mathcal{H}}, \qquad \forall\, (x, h) \in \mathcal{X} \times \mathcal{H},$$

where the *kernel sections* K_x are defined as

$$K_x(y) = K(x, y), \qquad \forall y \in \mathcal{X}.$$

The reproducing property states that the representers of point-wise evaluation functionals coincide with the kernel sections. Starting from the reproducing

property, it is also easy to show that the representer of any bounded linear functional L is a function $K_L \in \mathcal{H}$ such that

$$K_L(x) = LK_x, \qquad \forall x \in \mathcal{X}.$$

Therefore, in a RKHS, the representer of any bounded linear functional can be obtained explicitly in terms of the reproducing kernel.

With reference to the deconvolution problem (1), we are interested in estimating functions defined over the time set $\mathcal{X} = \mathcal{T}$. By expressing the representers in terms of the kernel, the optimal solution (2) can be rewritten in the explicit form

$$h(t) = \sum_{i=1}^{\ell} c_i (u * K_t)(t_i).$$

Finally, the entries of the kernel matrix $\mathbf{K}$ can be expressed as

$$\mathbf{K}_{ij} = \int_{\mathcal{T}} \int_{\mathcal{T}} u(t_i - \tau_1) u(t_j - \tau_2) K(\tau_1, \tau_2) d\tau_1 d\tau_2.$$

3 Basic properties of the impulse response

By searching the impulse response in a RKHS, we are automatically assuming that h is point-wise well-defined and bounded over compact time sets. In this section, we show how several other important properties of the impulse response can be enforced by simply designing a suitable kernel function.

3.1 Causality

A dynamical system is said to be causal if the value of the output signal at a certain time instant T does not depend on values of the input in the future (for $t > T$). Causality is a prior knowledge that is virtually always incorporated in the model of a dynamical system. This is done already when the signals are classified as inputs or outputs of the system: the value of the output signals at a certain time is not allowed to depend on the values of the input signals in the future, whereas the other way around is possible.

For a LTI system, causality is equivalent to vanishing of the impulse response for negative times, namely

$$h(t) = 0, \quad \forall t < 0, \quad \forall h \in \mathcal{H}. \tag{3}$$

The following Lemma characterizes those RKHS that contain causal impulse responses, with a simple condition on the kernel function.

Lemma 1 *The RKHS $\mathcal{H}$ contains only causal impulse responses if and only if the reproducing kernel satisfy*

$$K(t_1, t_2) = H(t_1)H(t_2)\widetilde{K}(t_1, t_2). \tag{4}$$

where $H(t)$ is the Heaviside function defined as

$$H(t) = \begin{cases} 1, & t \geq 0 \\ 0, & \textit{else} \end{cases}$$

and $\widetilde{K}$ is a kernel defined for non-negative times.

From the simple result of Lemma 1, we can already see that the kernels needed for modeling impulse responses of dynamical systems are quite different from the typical kernels used for curve fitting. In order to encode a "privileged" direction in the time flow, they have to be asymmetric on the real line, and can also be discontinuous.

3.2 Stability

System stability is an important prior information that should be always incorporated in any identification method, whenever available. Perhaps, the most intuitive notion of stability is the so called BIBO (Bounded Input Bounded Output) condition that can be expressed as

$$\|u\|_\infty < +\infty \Rightarrow \|y\|_\infty < +\infty,$$

where $\|\cdot\|_\infty$ denotes the L^∞ norm. Assuming BIBO stability entails that the output of the system to be identified cannot diverge when excited with a bounded input signal. It is well-known that, for a LTI system, BIBO stability is equivalent to integrability of the impulse response:

$$\left| \int_{\mathcal{T}} h(t)dt \right| < +\infty. \tag{5}$$

Since stability is very often known to be satisfied by the system under study, it is interesting to characterize those reproducing kernel Hilbert spaces that contain only integrable functions. The following Lemma gives a necessary and sufficient condition (see e.g. [3]):

Lemma 2 *The RKHS $\mathcal{H}$ is a subspace of $L^1(\mathcal{T})$ if and only if*

$$\int_{\mathcal{T}} \left| \int_{\mathcal{T}} K(t_1, t_2)h(t_1)dt_1 \right| dt_2 < +\infty, \quad \forall h \in L^\infty(\mathcal{T}).$$

We can talk about *stability of the kernel*, with reference to kernels that satisfy the conditions of Lemma 2. It can be easily verified that integrability of the kernel is a sufficient condition for $\mathcal{H}$ to be a subspace of $L^1(\mathcal{T})$.

Lemma 3 *If $K \in L^1(\mathcal{T}^2)$, then $\mathcal{H}$ is a subspace of $L^1(\mathcal{T})$.*

It is worth observing that the condition of Lemma 3 is also necessary for nonnegative-valued kernels (i.e. such that $K(t_1, t_2) \geq 0$, for all t_1, t_2), as it can be seen by simply setting $f = 1$ in Lemma 2.

3.3 Delay

In view of causality, the value of the output signal at a certain time doesn't depend on the values of the input signal in the future. Let D denote the smallest time instant such that $h(t) \neq 0$:

$$D := \inf \{t \in \mathcal{T} : h(t) \neq 0\} .$$

By causality, D has to be nonnegative. If it is strictly positive, then the system is said to exhibit an input-output *delay* equal to D, meaning that $y(\tau)$ does not depend on $u(t)$ for any $t > \tau - D$. Once again, the prior knowledge of the delay D can be easily incorporated in the kernel function.

Lemma 4 *Every impulse response $h \in \mathcal{H}$ have a delay equal to D if and only if the reproducing kernel is in the form*

$$K_D(t_1, t_2) = K(t_1 - D, t_2 - D),$$

with K in the form (4).

If the value of D is unknown in advance, it can be treated as an hyper-parameter to be estimated from the data.

4 Kernels for continuous-time systems

In this section, we focus on some properties of continuous-time systems ($\mathcal{T} = \mathbb{R}$), such as smoothness of the impulse response and relative degree, and discuss how to enforce them by choosing suitable kernels.

4.1 Smoothness

Consider a continuous-time LTI system without delay (the delayed case can be simply handled via the change of variable discussed in Lemma 4). Impulse responses for continuous-time dynamical systems are typically assumed to have

some degree of smoothness. Smoothness can be expressed in terms of continuity of h and a certain number of time derivatives, everywhere with the possible exception of the origin. Regularity of the impulse response at $t = 0$ is related to the concept of *relative degree*, which is important enough to deserve an independent treatment (see the next subsection). Impulse responses with a high number of continuous derivatives corresponds to low-pass dynamical systems that attenuates high frequencies of the input signal. It is known, see e.g. [14], that regularity of the kernel propagates to every function in the RKHS. Therefore, prior knowledge about smoothness of the impulse response can be directly expressed in terms of the kernel function.

Lemma 5 *Let $\mathcal{H}$ denote a RKHS associated with the kernel in the form (4) with $\mathcal{T} = \mathbb{R}$. If $\widetilde{K}$ is k-times continuously differentiable on $(0, +\infty)^2$, then the restriction of every function $h \in \mathcal{H}$ to $(0, +\infty)$ is k-times continuously differentiable. In addition, point-wise evaluated derivatives are continuous linear functionals, i.e. for all $t > 0$ and $i \leq k$, there exists $C < +\infty$ such that*

$$|h^{(i)}(t)| \leq C\|h\|_{\mathcal{H}}, \quad \forall h \in \mathcal{H}.$$

4.2 Relative degree

The *relative degree* is an important concept for continuous-time dynamical systems. In many cases, prior knowledge about the relative degree is available thanks to simple physical considerations. The relative degree of an LTI system is directly linked to the regularity of the impulse response at $t = 0$ (or $t = D$ in the delayed case). In view of Lemma 4, all the left derivatives of the impulse response (with the convention $h^{(0)} = h$) have to vanish:

$$h^{(k)}(0^-) = 0, \quad \forall k \geq 0.$$

On the other hand, the right derivatives may well be different from zero. Assuming existence of all the necessary derivatives, the relative degree of a LTI system is a natural number k such that

$$h^{(i)}(0^+) = 0, \quad \forall i < k, \quad h^{(k)}(0^+) \neq 0.$$

If $h^{(i)}(0^+) = 0$ for all i, the relative degree is *undefined*.

If the relative degree is k, then the k-th derivative of the impulse response in $t = 0$ is discontinuous. Let's represent the impulse response in the form $h(t) = H(t)h_+(t)$, where H is the Heaviside step function, and assume that $h_+(t)$ admits at least k right derivatives at $t = 0$. By using distributional

derivatives and properties of the convolution, we have

$$
\begin{aligned}
y^{(k+1)}(t) &= (h^{(k+1)} * u)(t) \\
&= h^{(k)}(0^+)\left(\delta_0 * u\right)(t) + \int_{-\infty}^{t} u(\tau) h_+^{(k+1)}(t-\tau) d\tau \\
&= h^{(k)}(0^+) u(t) + \int_{-\infty}^{t} u(\tau) h_+^{(k+1)}(t-\tau) d\tau .
\end{aligned}
$$

The $(k+1)$-th time derivative of the output is the first derivative that is *directly* influenced by the input $u(t)$. Therefore, the system exhibits an input-output integral effect equivalent to a chain of k integrators on the input of a system with relative degree one.

Prior knowledge about the relative degree of the system can be enforced by designing the kernel according with the following Lemma.

Lemma 6 *Under the assumptions of Lemma 5, every impulse response $h \in \mathcal{H}$ has relative degree greater or equal than k if and only if*

$$\forall t \in \mathbb{R}, \quad \lim_{\tau \to 0^+} \widetilde{K}_t^{(i)}(\tau) = 0, \quad \forall i < k. \tag{6}$$

Hence, when the impulse response is searched within an RKHS, the relative degree of the identified system is directly related to the simple property (6) of the kernel function. We can therefore introduce the concept of *relative degree of the kernel*.

4.3 Examples

The simplest possible kernel for impulse response identification is perhaps the *Heaviside kernel*:

$$K(t_1, t_2) = H(t_1) H(t_2),$$

whose associated RKHS contains only step functions. This kernel has relative degree equal to one and is clearly not stable. As a second example, consider the *exponential kernel*

$$K(t_1, t_2) = H(t_1) H(t_2) e^{-\omega(t_1 + t_2)}. \tag{7}$$

This kernel is infinitely differentiable everywhere, except over the lines $t_1 = 0$ and $t_2 = 0$, where it is discontinuous. Since K is discontinuous, the relative degree is equal to one. The associated Hilbert space $\mathcal{H}$ contains exponentially decreasing functions.

5 Families of kernels for system identification

The exponential kernel defined in (7) satisfies the sufficient condition of Lemma 3, therefore the associated RKHS contains stable impulse responses of relative degree one (in fact, the space contains only stable exponential functions). Now, assume that a kernel K_1 with relative degree one is available. Then, we can easily generate a family of kernels of arbitrary relative degree via the following recursive procedure:

$$K_{i+1}(t_1, t_2) = \int_{-\infty}^{t_1} \int_{-\infty}^{t_2} K_i(\tau_1, \tau_2) d\tau_1 d\tau_2, \quad i \geq 1.$$

Unfortunately, the application of such procedure doesn't preserve integrability of the original kernel K_1. Consider for example the exponential kernel (7). Although K_1 is stable, all the other kernels K_i with $i \geq 2$ do not satisfy the necessary condition of Lemma 2 (to see this, it is sufficient to choose $h = 1$) and are therefore not BIBO stable. In the following, we describe some alternative ways of constructing families of stable kernels.

5.1 A family of stable kernels

In this subsection, we discuss a technique to construct stable kernels of any relative degree. The key idea to obtain stability is to introduce a change of coordinates that maps $\mathbb{R}^+$ into the finite interval $[0, 1]$, and then use a kernel over the unit square $[0, 1]^2$. Let $G : [0, 1]^2 \to \mathbb{R}$ denote a positive semidefinite kernel, and $h_\omega : \mathbb{R}^+ \to [0, 1]$ denote the exponential coordinate transformation

$$h_\omega(t) = e^{-\omega t}.$$

Then, we can construct a class of kernels defined as in (4), where

$$\widetilde{K}(t_1, t_2) = (t_1 t_2)^k \int_{\mathbb{R}^+} G(h_\omega(t_1), h_\omega(t_2)) d\mu(\omega), \quad k \in \mathbb{N}, \tag{8}$$

and μ is a probability measure. If $G(h_\omega(t_1), h_\omega(t_2))$ is a kernel with relative degree one, we can immediately check, using Lemma 6, that the kernel (8) has relative degree $k + 1$. To ensure BIBO stability, the mass of $\mu(\omega)$ should not be concentrated around zero and the kernel G must vanish sufficiently fast around the origin. The following Lemma gives a sufficient condition.

Lemma 7 *Let $G : [0, 1]^2 \to \mathbb{R}$ denote a kernel such that*

$$\frac{G(s_1, s_2)}{s_1 s_2} \leq C, \quad \forall (s_1, s_2) \in [0, 1]^2. \tag{9}$$

If the support of μ does not contains the origin, then the kernel (8) is BIBO stable for all $k \in \mathbb{N}$.

An example is the recently proposed *stable spline* kernel [5], obtained by choosing G as the cubic spline kernel (that can be also seen as the covariance function of an integrated Wiener process on $\mathbb{R}^+$):

$$G(s_1, s_2) = \frac{s_1 s_2 \min\{s_1, s_2\}}{2} - \frac{\min\{s_1, s_2\}^3}{6}.$$

A simple calculation shows that condition (9) is satisfied with $C = 1/3$. By using (8), we can start from this kernel to generate a class of stable kernels of arbitrary relative degree. For example, by choosing μ as the unit mass on a certain frequency $\omega > 0$, we obtain the kernel

$$\begin{aligned} K(t_1, t_2) &= (t_1 t_2)^k G(h_\omega(t_1), h_\omega(t_2)) \\ &= (t_1 t_2)^k \left(\frac{e^{-\omega(t_1 + t_2 + \max\{t_1, t_2\})}}{2} - \frac{e^{-3\omega \max\{t_1, t_2\}}}{6} \right), \end{aligned}$$

which is stable and has relative degree equal to $k + 1$.

5.2 Kernels for relaxation systems

Many real-world systems, such as reciprocal electrical networks whose energy storage elements are of the same type, or mechanical systems in which inertial effects may be neglected have the property that the impulse response never exhibits oscillations. *Relaxation systems*, see e.g. [18], are dynamical systems whose impulse response is a so-called *completely monotone* function. An infinitely differentiable function $f : \mathbb{R}^+ \to \mathbb{R}$ is called completely monotone if

$$(-1)^n f^{(n)}(t) \geq 0, \quad \forall n \in \mathbb{N}, \quad t > 0.$$

The following characterization of on completely monotone functions [2, 17] is a convenient tool that allows to generalize the simple exponential kernel defined in (7).

Theorem 1 (Bernstein-Widder) *An infinitely differentiable real-valued function f defined on the real line is completely monotone if and only if there exists a non-negative finite Borel measure μ on $\mathbb{R}^+$ such that*

$$f(t) = \int_{\mathbb{R}^+} e^{-t\omega} d\mu(\omega).$$

In view of this last theorem, completely monotone functions are characterized as mixture of decreasing exponentials or, in other words, as Laplace transforms of non-negative measures. Let f denote a completely monotone function, and consider the family of functions of the form

$$K(t_1, t_2) = H(t_1) H(t_2) f(t_1 + t_2). \tag{10}$$

By Theorem 1, we can easily verify that (10) defines a positive semidefinite kernel:

$$\sum_{i=1}^{\ell}\sum_{j=1}^{\ell} c_i c_j K(t_i,t_j) =$$
$$= \sum_{i=1}^{\ell}\sum_{j=1}^{\ell} c_i c_j H(t_i)H(t_j)\int_{\mathbb{R}^+} e^{-t_i\omega}e^{-t_j\omega}d\mu(\omega)$$
$$= \int_{\mathbb{R}^+}\left(\sum_{i=1}^{\ell} c_i H(t_i)e^{-t_i\omega}\right)^2 d\mu(\omega) \geq 0.$$

Clearly, not every function in the associated RKHS is a completely monotone impulse response. However, all the kernel sections K_t are completely monotone.

Now, observe that, unless $f = 0$, the relative degree of kernel (10) is always one. Indeed, if the relative degree is greater than one, then we have

$$f(t) = K_t(0^+) = 0, \quad \forall t \in \mathbb{R}.$$

By using Lemma 3, we can check that, when the support of μ does not contain the origin, the kernel (10) is BIBO stable:

$$\int_{\mathbb{R}^2} |K(t_1,t_2)|\, dt_1 dt_2 =$$
$$= \int_{\epsilon}^{+\infty}\left(\int_{\mathbb{R}^+} e^{-t_1\omega}dt_1\right)\left(\int_{\mathbb{R}^+} e^{-t_2\omega}dt_2\right)d\mu(\omega)$$
$$= \int_{\epsilon}^{+\infty}\frac{d\mu(\omega)}{\omega^2} \leq \frac{1}{\epsilon^2}\int_{\epsilon}^{+\infty} d\mu(\omega) < +\infty.$$

On the other hand, if the support of μ contains the origin, we may obtain unstable kernels. For instance, when μ is the unitary mass centered in the origin, we obtain the Heaviside kernel $H(t_1)H(t_2)$, which is not stable. Finally, observe that not all the kernels of the form (10) that vanishes when t_1 or t_2 tend to $+\infty$ are stable, as shown by the following counterexample:

$$K(t_1,t_2) = \frac{H(t_1)H(t_2)}{1+(t_1+t_2)^2}.$$

This kernel is indeed of the type (10), since the function $(1+t^2)^{-1}$ is completely monotone. However, the necessary condition of Lemma 2 is not satisfied with

$h = 1$:

$$\int_{\mathbb{R}^+} \left| \int_{\mathbb{R}^+} \frac{dt_1}{1+(t_1+t_2)^2} \right| dt_2$$
$$= \int_{\mathbb{R}^+} \left(\frac{\pi}{2} - \tan^{-1}(t_2) \right) dt_2$$
$$= t_2 \left(\frac{\pi}{2} - \tan^{-1}(t_2) \right) \Big|_0^{+\infty} + \int_{\mathbb{R}^+} \frac{t_2}{1+t_2^2} dt_2$$
$$= 1 + \frac{1}{2} \log(1+t_2^2) \Big|_0^{+\infty} = +\infty.$$

5.3 Translation invariant kernels are not stable

In contrast with (10), consider now kernels the type

$$K(t_1, t_2) = H(t_1)H(t_2)f(t_1 - t_2). \tag{11}$$

The following classical result [13] characterizes the class of functions such that $f(t_1 - t_2)$ is a positive semidefinite kernel.

Theorem 2 (Schoenberg) *Let $f : \mathbb{R} \to \mathbb{R}$ denote a continuous function. Then, $f(t_1 - t_2)$ is a positive semidefinite kernel if and only if there exists a non-negative finite Borel measure μ on $\mathbb{R}^+$ such that*

$$f(t) = \int_{\mathbb{R}^+} \cos(t\omega) d\mu(\omega).$$

Hence, when f is the cosine transform of a non-negative measure, the functions of the form (11) are positive semidefinite kernels, since they are the product of the Heaviside kernel and a positive semidefinite kernel. The family includes oscillating functions of the type $f(t_1 - t_2) = \sum_i d_i \cos(\omega_i(t_1 - t_2))$ that are apparently instable, but also widely used kernels like the Gaussian

$$f(t_1 - t_2) = e^{-\omega(t_1 - t_2)^2}.$$

In view of their popularity, one might be tempted to adopt these kernels for system identification. However, a simple calculation shows that, unless $f = 0$, kernels of the type (11) are never stable:

$$\int_{\mathbb{R}^+} \left| \int_{\mathbb{R}^+} f(|t_1 - t_2|) dt_1 \right| dt_2 =$$
$$= \int_{\mathbb{R}^+} \left| \int_{-t_2}^{+\infty} f(|\tau|) d\tau \right| dt_2$$
$$\geq \int_{\mathbb{R}^+} \left| \int_{\mathbb{R}^+} f(|\tau|) d\tau \right| dt_2$$
$$= \left| \int_{\mathbb{R}^+} f(|\tau|) d\tau \right| \int_{\mathbb{R}^+} dt_2 = +\infty.$$

Since BIBO stability is almost always satisfied in real world systems, the class of kernels (11) seems to be not well suited for system identification.

6 Conclusions

Regularization techniques in RKHS are flexible tools for both discrete and continuous time LTI system identification. They allow to handle datasets with sparse temporal sampling while incorporating several types of prior informations. As shown in this report, informations such as BIBO stability, relative degree, and smoothness can be encoded by designing simple properties of the kernel function. We have also discussed several examples of kernels, showing that some of them are well-suited to describe stable dynamics while others are not.

Appendix

Proof of Lemma 1: By the reproducing property, we have

$$h(t) = \langle K_t, h \rangle_{\mathcal{H}},$$

If the kernel K satisfies the condition of the Lemma, we have $K_t = 0$ for $t < 0$, so that $h(t)$ equals zero for negative t. On the other hand, since $K_t \in \mathcal{H}$ for all t, condition (3) implies

$$K_t(\tau) = K(t, \tau) = 0, \quad \forall t < 0, \quad \forall \tau \in \mathcal{T}.$$

In view of symmetry, it follows that the kernel must necessarily be in the form defined by the Lemma.

$\square$

Proof of Lemma 2:

This is an immediate corollary of Proposition 4.2. in [3].

$\square$

Proof of Lemma 3:

If K is integrable, for all $h \in L^\infty(\mathcal{T})$, we have

$$\begin{aligned} &\int_{\mathcal{T}} \left| \int_{\mathcal{T}} K(t_1, t_2) h(t_1) dt_1 \right| dt_2 \\ &\leq \|h\|_\infty \int_{\mathcal{T}} \int_{\mathcal{T}} |K(t_1, t_2)| dt_1 dt_2 < +\infty. \end{aligned}$$

$\square$

Proof of Lemma 4: The proof is similar to that of Lemma 1. By the reproducing property, we have

$$h(t) = \langle K_t, h \rangle_{\mathcal{H}},$$

If the kernel K satisfies the condition of the Lemma, we have $K_t = 0$ for $t < D$, so that $h(t)$ equals zero $t < D$. On the other hand, since $K_t \in \mathcal{H}$ for all t, condition (3) implies

$$K_t(\tau) = K(t, \tau) = 0, \quad \forall t < D, \quad \forall \tau \in \mathcal{T}.$$

In view of symmetry, it follows that the kernel must necessarily be in the form defined by the Lemma.

$\square$

Proof of Lemma 5:

The restriction of the kernel to $(0, +\infty)^2$ is k-times continuously differentiable. Then, by Corollary 4.36 of [14], it follows that the restriction of every function $h \in \mathcal{H}$ to the interval $(0, +\infty)$ is k-times continuously differentiable, and point-wise evaluated derivatives are bounded linear functionals.

$\square$

Proof of Lemma 6:

In view of Lemma 5, point-wise evaluated derivatives at any $t > 0$ are bounded linear functionals. By the reproducing property, we have

$$\lim_{\tau \to 0^+} h^{(i)}(\tau) = \lim_{\tau \to 0^+} \langle K_\tau^{(i)}, h \rangle_{\mathcal{H}}.$$

If all the impulse responses $h \in \mathcal{H}$ have relative degree greater or equal than k, the left hand side is zero for all $h \in \mathcal{H}$ and $i < k$. It follows that

$$\lim_{\tau \to 0^+} K_\tau^{(i)} = 0, \quad \forall i < k.$$

Condition (6) follows from the symmetry of the kernel. Conversely, if condition (6) holds, we immediately obtain

$$\lim_{\tau \to 0^+} h^{(i)}(\tau) = 0, \quad \forall i < k, \quad \forall h \in \mathcal{H},$$

since the inner product is continuos. It follows that the relative degree of any function h of the space is greater or equal than k.

$\square$

Proof of Lemma 7: We have

$$
\begin{aligned}
&\int_{\mathbb{R}^2} |K(t_1,t_2)|\, dt_1 dt_2 = \\
&= \int_{\mathbb{R}^+} \int_{\mathbb{R}^+ \times \mathbb{R}^+} (t_1 t_2)^k \left| G(h_\omega(t_1), h_\omega(t_2)) \right| dt_1 dt_2 d\mu(\omega) \\
&= \int_{[0,1]^2} (\ln s_1 \ln s_2)^k \left| G(s_1,s_2) \right| ds_1 ds_2 \int_{\mathbb{R}^+} \frac{d\mu(\omega)}{\omega^{2(1+k)}} \\
&\leq \frac{1}{\epsilon^{2(1+k)}} \int_{[0,1]^2} s_1 s_2 \left(\ln s_1 \ln s_2\right)^k \frac{|G(s_1,s_2)|}{s_1 s_2} ds_1 ds_2 \\
&\leq \frac{C}{\epsilon^{2(1+k)}} \int_{[0,1]^2} s_1 s_2 \left(\ln s_1 \ln s_2\right)^k ds_1 ds_2 \\
&\leq \frac{C}{\epsilon^{2(1+k)}} e^{-2k(1-\log k)} < +\infty.
\end{aligned}
$$

The thesis follows by applying Lemma 3.

□

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